Properties of Regular Polygons

Polygon

A polygon is a plane shape (two-dimensional) with straight sides. Examples include triangles, quadrilaterals, pentagons, hexagons and so on.

Regular

A "Regular Polygon" has:

- all sides equal and
- all angles equal.

Otherwise it is irregular.

Here we look at Regular Polygons only.

Properties

So what can we know about regular polygons? First of all, we can work out angles.

Exterior Angle

The Exterior Angle is the angle between any side of a shape, and a line extended from the next side.

All the Exterior Angles of a polygon add up to 360°, so:

Each exterior angle must be $360°/n$
Example: What is the exterior angle of a regular octagon?

An octagon has 8 sides, so:

\[
\text{Exterior angle} = \frac{360°}{n} = \frac{360°}{8} = 45°
\]

Interior Angles

The \textit{Interior Angle} and Exterior Angle are measured from the same line, so they \textbf{add up to 180°}.

\[
\text{Interior Angle} = 180° - \text{Exterior Angle}
\]

We know the \textbf{Exterior angle} = \frac{360°}{n}, so:

\[
\text{Interior Angle} = 180° - \frac{360°}{n}
\]
Which can be rearranged like this:

\[
\text{Interior Angle} = 180^\circ - \frac{360^\circ}{n} = (n \times 180^\circ / n) - (2 \times 180^\circ / n) = (n-2) \times \frac{180^\circ}{n}
\]

So we also have this:

\[
\text{Interior Angle} = (n-2) \times \frac{180^\circ}{n}
\]

**Example: What is the interior angle of a regular octagon?**

A regular octagon has 8 sides, so:

\[
\text{Exterior Angle} = \frac{360^\circ}{8} = 45^\circ
\]

\[
\text{Interior Angle} = 180^\circ - 45^\circ = 135^\circ
\]

Or we could use:

\[
\text{Interior Angle} = (n-2) \times \frac{180^\circ}{n} = (8-2) \times \frac{180^\circ}{8} = 6 \times \frac{180^\circ}{8} = 135^\circ
\]

**Example: What are the interior and exterior angles of a regular hexagon?**

A regular hexagon has 6 sides, so:
Exterior Angle = $360^\circ / 6 = 60^\circ$

Interior Angle = $180^\circ - 60^\circ = 120^\circ$

And now for some names:

"Circumcircle, Incircle, Radius and Apothem ..."

Sounds quite musical if you repeat it a few times, but they are just the names of the "outer" and "inner" circles (and each radius) that can be drawn on a polygon like this:

The "outside" circle is called a **circumcircle**, and it connects all vertices (corner points) of the polygon.

The radius of the circumcircle is also the **radius** of the polygon.

The "inside" circle is called an **incircle** and it just touches each side of the polygon at its midpoint.

The radius of the incircle is the **apothem** of the polygon.

(Not all polygons have those properties, but triangles and regular polygons do).

**Breaking into Triangles**

We can learn a lot about regular polygons by breaking them into triangles like this:

Notice that:

- the "base" of the triangle is one side of the polygon.
- the "height" of the triangle is the "Apothem" of the polygon

Now, the **area of a triangle** is half of the base times height, so:

\[
\text{Area of one triangle} = \text{base} \times \text{height} / 2 = \text{side} \times \text{apothem} / 2
\]

To get the area of the whole polygon, just add up the areas of all the little triangles ("n" of them):

\[
\text{Area of Polygon} = n \times \text{side} \times \text{apothem} / 2
\]

And since the perimeter is all the sides = \(n \times \text{side}\), we get:

\[
\text{Area of Polygon} = \text{perimeter} \times \text{apothem} / 2
\]

### A Smaller Triangle

By cutting the triangle in half we get this:

(Note: The angles are in **radians**, not **degrees**)

The small triangle is right-angled and so we can use **sine, cosine and tangent** to find how the side, radius, apothem and "n" are related:

\[
\sin(\pi/n) = (\text{Side}/2) / \text{Radius} \quad \Rightarrow \quad \text{Side} = 2 \times \text{Radius} \times \sin(\pi/n)
\]

\[
\cos(\pi/n) = \text{Apothem} / \text{Radius} \quad \Rightarrow \quad \text{Apothem} = \text{Radius} \times \cos(\pi/n)
\]
Regular Polygons - Properties

\[
\tan(\pi/n) = \frac{\text{Side}/2}{\text{Apothem}} \quad \Rightarrow \quad \text{Side} = 2 \times \text{Apothem} \times \tan(\pi/n)
\]

There are a lot more relationships like those (most of them just "re-arrangements"), but those will do for now.

More Area Formulas

We can use that to calculate the area when we only know the Apothem:

**Area of the Small Triangle** = \( \frac{1}{2} \times \text{Apothem} \times (\text{Side}/2) \)

And we know (from the "tan" formula above) that:

\[
\text{Side} = 2 \times \text{Apothem} \times \tan(\pi/n)
\]

So:

**Area of Small Triangle** = \( \frac{1}{2} \times \text{Apothem} \times (\text{Apothem} \times \tan(\pi/n)) \)

= \( \frac{1}{2} \times \text{Apothem}^2 \times \tan(\pi/n) \)

And there are 2 such triangles per side, or 2\(n\) for the **whole polygon**:

**Area of Polygon** = \( n \times \text{Apothem}^2 \times \tan(\pi/n) \)

When we don't know the Apothem, we can use the same formula but re-worked for Radius or for Side:

**Area of Polygon** = \( \frac{1}{2} \times n \times \text{Radius}^2 \times \sin(2 \times \pi/n) \)

**Area of Polygon** = \( \frac{1}{4} \times n \times \text{Side}^2 / \tan(\pi/n) \)

A Table of Values

And here is a table of Side, Apothem and Area compared to a Radius of "1", using the formulas we have worked out:
### Regular Polygons - Properties

<table>
<thead>
<tr>
<th>Type</th>
<th>Name when Regular</th>
<th>Sides (n)</th>
<th>Shape</th>
<th>Interior Angle</th>
<th>Radius</th>
<th>Side</th>
<th>Apothem</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Triangle</strong></td>
<td>Equilateral Triangle</td>
<td>3</td>
<td>△</td>
<td>60°</td>
<td>1.732</td>
<td>0.5</td>
<td>1.2</td>
<td>(3√3)</td>
</tr>
<tr>
<td><strong>Quadrilateral</strong></td>
<td>Square</td>
<td>4</td>
<td>□</td>
<td>90°</td>
<td>1.414</td>
<td>0.707</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td><strong>Pentagon</strong></td>
<td>Regular Pentagon</td>
<td>5</td>
<td>□</td>
<td>108°</td>
<td>1.176</td>
<td>0.809</td>
<td>2.3</td>
<td></td>
</tr>
<tr>
<td><strong>Hexagon</strong></td>
<td>Regular Hexagon</td>
<td>6</td>
<td>□</td>
<td>120°</td>
<td>0.866</td>
<td>0.924</td>
<td>2.8</td>
<td>(2√3)</td>
</tr>
<tr>
<td><strong>Heptagon</strong></td>
<td>Regular Heptagon</td>
<td>7</td>
<td>□</td>
<td>128.571°</td>
<td>0.868</td>
<td>0.901</td>
<td>2.7</td>
<td></td>
</tr>
<tr>
<td><strong>Octagon</strong></td>
<td>Regular Octagon</td>
<td>8</td>
<td>□</td>
<td>135°</td>
<td>0.765</td>
<td>0.924</td>
<td>2.8</td>
<td></td>
</tr>
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<td>...</td>
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<td>...</td>
</tr>
<tr>
<td><strong>Pentacontagon</strong></td>
<td>Regular Pentacontagon</td>
<td>50</td>
<td>□</td>
<td>172.8°</td>
<td>0.126</td>
<td>0.998</td>
<td>3.1</td>
<td></td>
</tr>
</tbody>
</table>

(Note: values correct to 3 decimal places only)

### Graph

And here is a graph of the table above, but with number of sides ("n") from 3 to 30.

Notice that as "n" gets bigger, the Apothem is tending towards 1 (equal to the Radius) and that the Area is tending towards \( \pi = 3.14159... \), just like a circle.

What is the Side length tending towards?